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Reduced Order Models and Control of Large Scale Aero-Elastic Simulations

by
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ABSTRACT

Control of large-scale, aero-elastic models requires advanced model reduction techniques for implementation of feedback control. New reduced order model techniques must be developed based on the concepts and physics of fluid-structure interaction. Current methods are inefficient and inaccurate when dealing with these large scale aero-elastic models. Reduced order model (ROM) based controllers may produce adverse affects on un-modeled modes causing instability in the system. The idea of compensation is introduced to correct for this problem. The primary goal of this paper is to explore three separate techniques for developing ROM based feedback controllers to aero-elastic systems. They are the following: modal truncation with residual mode filter (RMF) compensation, Schur form with residual state filter (RSF) compensation and the singular perturbation approach.

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by

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Table of Contents

Nomenclature

$\underline{x}$	General State Vector
x_N	Reduced Order Model States
x_R	Residual States
x_Q	States Forced Unstable by ROM Controller
x_S	Structural States
x_F	Fluid States
$\underline{u}$	Actuated Input
$\underline{y}$	Sensor Output
K	Observer Gain Matrix
G	Controller Gain Matrix
L	Observer / Controller Combined Matrix

List of Figures

1 Introduction

In recent years, the improvements in computational capabilities and numerical algorithms have allowed for the design and formation of highly accurate, physics-based, non-linear, large-scale aero-elastic systems(1). An aero-elastic model is a model which describes the interaction between structural elasticity and fluid aerodynamics. Complex finite element analysis (FEA) and computational fluid dynamics (CFD) are used to build these models. The fluid and structure are broken up into meshes and nodes which describe the state of the body of the structure and state of the flow field of the fluid. Dynamical equations, such as conservation of mass, momentum, and energy exist at each of these nodes. Computational programs are able to calculate unknown variables (stresses, deflections, forces, pressures, etc) at each of these nodes for given set of boundary conditions.

Fluid systems need an extremely large number of nodes to define an accurate mesh to model the fluid dynamics. These models may have around a million degrees of freedom; systems of this scale can typically have matrices on the size of 10,000 x 10,000(1). Designing control systems for this scale is not feasible; reduced order models (ROM) are needed to capture the dynamic behavior of the large scale model so that feedback control can be implemented. Current methods are inefficient and inaccurate for reducing such large scale models. New methods, or combinations of methods, are needed to accurately reduce the order of the model so that ROM based control is feasible.

This paper will address three separate techniques for reduced order modeling of large scale aero-elastic systems: modal truncation with residual mode filter (RMF) compensation, Schur decomposition with residual state filter (RSF) compensation, and singular perturbation approach.

The first method, modal or eigen truncation, is a widley used technique for creating a ROM. The system is diagonalized and the lower / most dominant modes are retained for the ROM. The lower frequency modes are normally the most dominante modes because they have the highest displacement and highest energy states. This method is very appealing for control of structure-space systems because structures can be very well modeled by only a few modes (2). These modes will be the lowest frequency, most dominant modes of the system. It is also fairly simple to numerically calculate eigenvectors and eigenvalues of a system with today's computational techniques and algorithms. The downfall of this method is that when applied to an aero-elastic model, fluid eigenvalues tend to be very closely packed which requires for a large number of retained eigenvalues. This will limit the size of the reduced order model, and makes it hard to determine which fluid eigenvalues are most dominant.

The second method, Schur decomposition (6), is very similar to that of modal truncation. The only difference between the two methods is that the system does not need to be diagonalized in the Schur decomposition. The linear time invariant (LTI) system can be decomposed using Schur's algorithm so that the system is in upper triangular block form. This method is appealing to ROM designers because it is merely a rotation through a unitary matrix, which is a numerically reasonable task to perform, which will also incur less round off and computation error when compared to diagonalization methods. The only difficulty with this method is that spillover occurs in the off diagonal blocks. Spillover is the interaction between the states through the upper right matrix block in Schur form. However, there are techniques to compensate for this affect.

Once the reduced order model is formulated, a controller can be designed about the ROM using standard techniques. The ROM controller may inadvertently cause some un-modeled states to go unstable. This is where compensation comes into play. Implementation of residual mode / residual state filtering (RMF / RSF) is a simple idea. The filter will basically cancel out the effect of these problematic states. This will be discussed in further detail in a later section.

The third technique, singular perturbation (8), attempts to organize the model into slow and fast states, or structure and fluid states, respectively. The fluid is then reduced by this perturbation, which basically assumes that the fluid's derivative will go to zero, making it infinitely faster than the structure. A ROM can then be designed based on this idea. It will be shown that this perturbation in aero-elastic simulations can be then related to the velocity or Reynolds number of the fluid flow.

1.1 Background

The model used for the implementation of the different reduced order model techniques is an F-16, high performance air vehicle, model obtained from Charbel Farhat (1). This F-16 aero-elastic model contains both structural and fluid states. The model has a total of 78 states, 18 of which are structural states and the other 60 are fluid states. The model has already been greatly reduced for purposes other than control. The objective of the model was to be able substitute different flight criteria to see if flutter occurred in the system. Flutter exists in all structures and is merely a vibration which becomes amplified through interactions with external aerodynamic forces. Flutter can be disastrous in high performance aircraft, which spawns the need for a control system that will be able to reduce or dampen the flutter affects. The goal of this research project was to implement these various techniques for designing ROM feedback controllers on this relatively large model.

The structural representation of the F-16 model is based on the normal mass (M), spring (k) and damper (D) dynamical system:

$$M\ddot{u} + D\dot{u} + ku = p_\infty P w \tag{1.1.1}$$

The right hand of the equation is merely a forcing function based on the fluid states (w) and multiplied by the freestream pressure (p_∞) with a coordinate transformation (P). The terms ($u, \dot{u}, \ddot{u}$) are the position, velocity and acceleration respectively. The fluid dynamics of the model were formulated by proper orthogonal decomposition (POD) and looks like the following:

$$\underline{\dot{w}} + \sqrt{\frac{p_\infty}{\rho_\infty}} H\underline{w} + L\underline{\dot{u}} + \sqrt{\frac{p_\infty}{\rho_\infty}} N\underline{\mathrm{u}} = 0 \tag{1.1.2}$$

The fluid matrices (P, H, L, and N) were formed by complex means of computation fluid dynamics (CFD). They appear as a large assortment of numbers that lack physical meaning; conversely, for the structural system, each block represents a physical term such as damping, frequency, etc. ρ_∞ is the freestream air density. The fluid and structural equations can be combined in following way:

$$\begin{bmatrix} \dot{w} \\ \dot{u} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{P_\infty}{\rho_\infty}} H & -L & -\sqrt{\frac{P_\infty}{\rho_\infty}} \mathrm{N} \\ 0 & 0 & I \\ \mathrm{P}_\infty \mathrm{P} & -\Omega^2 & D \end{bmatrix} \begin{bmatrix} w \\ u \\ \dot{u} \end{bmatrix} \tag{1.1.3}$$

The state vector (x) is then defined to be

$$x \equiv \begin{bmatrix} w \\ u \\ \dot{u} \end{bmatrix} \quad \text{then} \quad \dot{x} \equiv \begin{bmatrix} \dot{w} \\ \dot{u} \\ \ddot{u} \end{bmatrix} \tag{1.1.4}$$

The system is now a linear time invariant (LTI) homogeneous differential equation in the form, $\dot{x} = Ax$. The next major task is to devise the sensor and actuator matrices so that feedback control can be performed. Because this model was purely an open loop system used for flutter analysis at different flight conditions, these matrices did not exist.

It was our initial primary task to decide upon what the actuation and sensor matrices should be. Actuation in the model was set up so that one could apply a normal force to a certain node on the wing. This actuation mechanism is not completely correct when thinking about in-flight dynamics. The actuators (flaps, rudder, ailerons, etc.) will move according to the pilot's commands and change the fluid flow, thereby producing new pressure gradients along the wing; thus, changing the lift, flight direction, or angle of attack of the aircraft. In future research it may be more accurate to view the fluid as the main actuator. Because no actuation is being done through the fluid, the fluid can be seen completely as a parasitic system, interacting with the structure independent of inputs and outputs.

The structural system is assumed to be fully actuated and sensed, that is identity matrices (I) are used in place of the actuating and sensing matrices. Only the position of the structural modes is being sensed. What this means is that a certain actuator would be able to entirely control a single mode of the structure without affecting any other modes. This is a rather large assumption to make, but we do it as a starting point for this kind of study. Even with this fully sensed / actuated model, problems still occurred with the reduced order model feedback controller. The actuator matrix (B) and the sensing matrix (C) can be then written as:

$$B = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \; and \; C = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$$

(1.1.5)

The entire system in first order linear form can be written as:

$$\begin{cases} \dot{x} \equiv \begin{bmatrix} \dot{w} \\ \dot{u} \\ \ddot{u} \end{bmatrix} = \overbrace{\begin{bmatrix} -\sqrt{\frac{P_\infty}{\rho_\infty}}H & -B & -\sqrt{\frac{P_\infty}{\rho_\infty}}C \\ 0 & 0 & I \\ \mathrm{P}_\infty \mathrm{P} & -\Omega^2 & D \end{bmatrix}}^{\mathrm{A}} \begin{bmatrix} w \\ u \\ \dot{u} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u \\ y = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \begin{bmatrix} w \\ u \\ \dot{u} \end{bmatrix} \end{cases} \tag{1.1.6}$$

Or

$$\begin{cases} \underline{\dot{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{cases} \tag{1.1.7}$$

The LTI state-space system is represented in block diagram form in Figure 1.

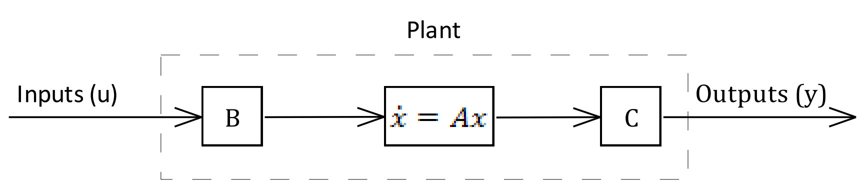

Figure 1: Block Diagram of Open-Loop State-Space System

The idea behind feedback control is to take the outputs from the sensors and relate them to the inputs via a controller algorithm. This will allow for the system to be able to react to its own dynamics. However, once feedback control is implemented problems may occur; for example, the controller may drive the system unstable. The control system designer's goal is to eliminate these instabilities and have the outputs converge to zero.

In open loop form the system's stability is determined by the eigenvalues of the A matrix. If one were to completely turn off the inputs and outputs of the system, the equation $\dot{x} = Ax$ would be left. This is a linear time-invariant (A does not depend on time) homogeneous differential equation. The homogenous solution (x(t)), or null space, of this differential equation is the matrix exponential, $x(t) = e^{At}c$. If A is a simple matrix, it is known that the eigenvectors (P) of A will diagonalize the matrix exponential of A, $x(t) = Pe^{\Lambda t}P^{-1}$, where Λ is a diagonal matrix of eigenvalues (λ). It can then be observed that if Real $(\lambda_k) < 0$ then $e^{\Lambda t}$ will not to blow up as $t \to \infty$. Therefore, the real part of the eigenvalues must be less strictly less than zero for stability to occur. If even one of the eigenvalues hits zero or becomes positive the system will become unstable. (3)

For the flight criteria $\sqrt{\frac{P_\infty}{\rho_\infty}} = 1$ the system in open loop form is barely stable. The open loop poles (eigenvalues) can be seen in Figure 2. It is observed that the open loop poles are all stable, even though some poles are just inside the left ½ plane. The eigenvalues of the fluid states lie along the real axis while the eigenvalues of the structure states appear in complex conjugate pairs. Note that the fluid poles are very densely packed while the structure poles are further separated. Generally, the fluid poles are faster (larger magnitude of the negative real part of the eigenvalue) than the structure poles, which supports the physical difference between solid (slow) and fluid (fast), which is used later on in the research, see section 2.3.

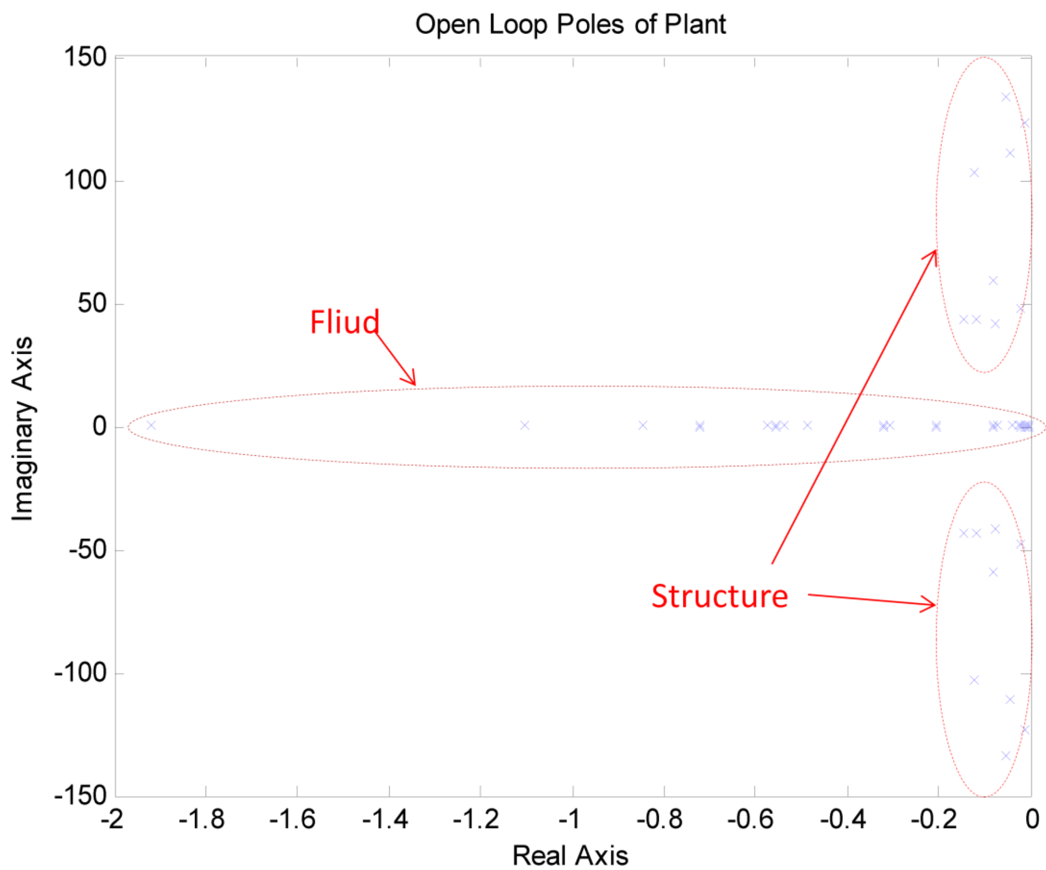

Figure 2: Open Loop Eigenvalues of Matrix A

1.2 Feedback Control Design

If all states were known and available at any time, full state feedback control would be easy to use. Wonham's theorem states the following:(4)

$$(A, B) Controllable \iff \exists\ G\ \ni (A + BG)\ has\ arbitrary\ poles$$

This theorem means that if control matrices A and B are controllable there exists a matrix G such that A+BG will have arbitrary pole placement. Arbitrary pole placement is important to the control designer so that he or she may place the closed loop poles wherever desired. The system would then look like the figure 3.

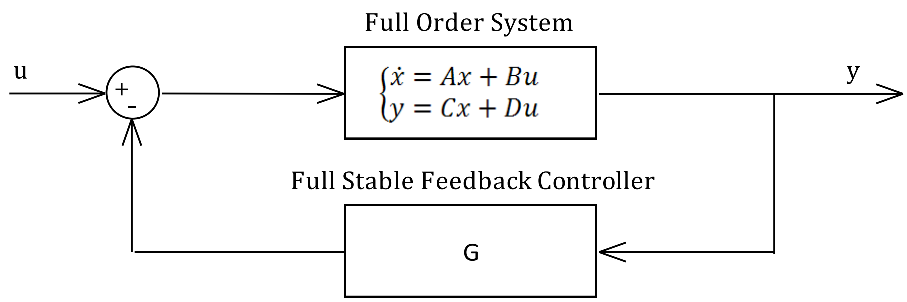

Figure 3: Diagram of Full Stable Feedback Controller

The system can then be written as the following dynamical equations

$$\begin{cases}\dot{x} = Ax + Bu \\ y = Cx + Du \\ u = Gx\end{cases} \tag{1.2.1}$$

Simply substitute the input (u) into the state space equations to for the closed loop system

$$\begin{cases}\dot{x} = (A + BG)x + Bu \\ y = (C + DG)x + Du\end{cases} \tag{1.2.2}$$

where the new A matrix (A+BG) has arbitrary pole placement. Therefore, this system can be stabilized under any conditions with the matrix G.

Since the states (x) are *NOT* known, this method will not entirely work. A state estimator needs to be designed and implemented along with the controller (G). The state estimator will be designed such that:

$$\begin{cases}\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}) \\ \hat{y} = C\hat{x} + Du\end{cases} \tag{1.2.3}$$

State estimation error is defined $e \equiv (\hat{x} - x) \underset{t\to\infty}{\longrightarrow} 0$.

The error dynamics can then be calculated as follows:

$$\dot{e} = (A - KC)e \tag{1.2.4}$$

It is now realized that the eigenvalues of (A-KC) must have real parts less than zero for the insured convergence of the state estimator. Wonhams theory also predicts the stability of the state estimator by: (4)

$$(A, C)Observable \Leftrightarrow \exists\, K \ni (A - KC)\ has\ arbitrary\ poles$$

The above equation indicates that, if (A,C) is observable, then there will exist a matrix K such that A-KC will have arbitrary pole placement. The two control theorems are then combined in the separation principle to introduce feedback control with state estimation such that:

$$\begin{cases} u = G\hat{x} \\ \dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}) \\ \hat{y} = Cx \end{cases} \tag{1.2.5}$$

The system will have arbitrary pole placement if and only if (A,B) controllable and (A,C) observable. Therefore, the poles may be placed wherever the designer chooses. This is an extremely powerful method for designing full state feedback control, but often the model will be too large to use the controller (1.2.5) and reduced order models are needed for control design and implementation.

2 Reduced Order Model Procedures

The following procedures are used to incorporate reduced order model based feedback control of aeroelastic systems. Each reduced order model technique has a unique procedure which will be outlined in the following sections. The three methods used were the following: modal truncation with residual mode filter (RMF) compensation, singular perturbation approach with balanced reduction, and upper triangularization with residual state filter (RSF) compensation. Only the general procedure of each method will be described and no specific results will be examined in these sections.

2.1 Modal Truncation with RMF Compensation

The aeroelastic model that contains both fluid states and structural states organized so that it is in first-order linear time invariant form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.1.1}$$

This system can be block-diagonalized using a coordinate transformation obtained from the eigenspaces of the A matrix. Many different computational methods exist to solve for the eigenvectors and eigenvalues of simple, non-singular matrices [ref]. The diagonalization can be done so the system is broken up into 3 subsystems x_N, x_R, and x_Q. The x_N subsystem will serve as the reduced order model from which the controller will be designed and is designated by the (A_N, B_N, C_N, D) matrices. It will look like equation (2.1.2) where A_N, A_R and A_Q can be diagonal matrices. For the moment, the Q and R subsystems will be combined; they are only the un-modeled modes.

$$\begin{cases} \dot{x}_N = A_N x_N \qquad\qquad\qquad + B_N u \\ \dot{x}_R = \qquad\quad A_R x_R \qquad\quad + B_R u \\ \dot{x}_Q = \qquad\qquad\qquad A_Q x_Q + B_Q u \\ y = C_N x_N + C_R x_R + C_Q x_Q + Du \end{cases} \tag{2.1.2}$$

A_N is chosen so that all of the *open loop instabilities* of the A matrix in equation (1) are contained in A_N. That is, all of the eigenvalues of A which have the real part greater than or equal to zero (unstable) will be selected as part of the x_N subspace. A_N will also include the dominant modes of the system, which are usually the lowest frequency modes of the structure.

Once the system is in diagonal linear form and the x_N subspace is defined, and a reduced order model based controller and state estimator are designed for the ROM system (A_N, B_N, C_N, D). The feedback controller and state estimator are shown in Figure 4.

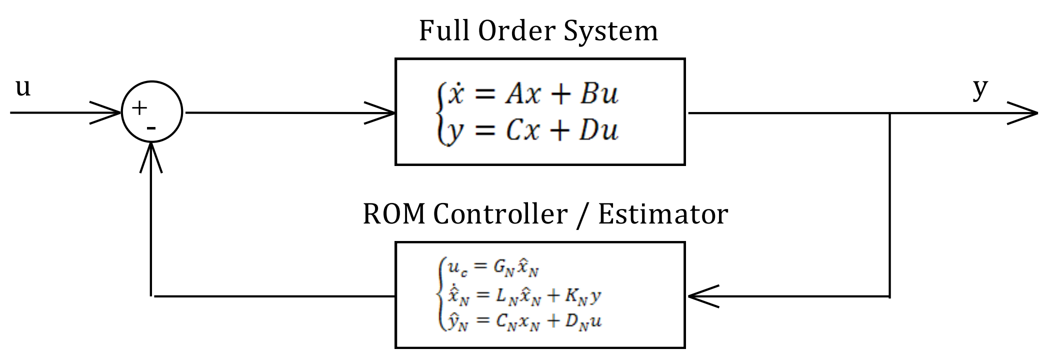

Figure 4: Reduced Order Model Feedback Controller

The controller and state estimator will be designed as discussed in section 1.2. Matrices are chosen such that the eigenvalues of (A_N+$B_N G_N$) and (A_N-$K_N C_N$) are all stable (real parts of eigenvalues less than zero). These will formulate the L, K, and G matrices.

$$\begin{cases} u_c = G_N \hat{x}_N \\ \dot{\hat{x}}_N = A_N \hat{x}_N + B_N u + K_N (y - \hat{y}_N) \\ \hat{y}_N = C_N x_N + Du \end{cases} \tag{2.1.3}$$

The margin of stability, or the distance of the spectrum from the imaginary axis, will allow for different design criteria. The margin of stability allows for a faster or slower controller.

Once the ROM based controller has been designed and implemented, the closed loop system can be formed. An error term (e_N) is defined such that the error is the difference between the estimated state and the actual state, as seen in equation (2.1.4).

$$e_N \equiv (\hat{x}_N - x_N) \tag{2.1.4}$$

Next, the error derivative is calculated to be:

$$\dot{e_N} \equiv \left(\dot{\hat{x}}_N - \dot{x}_N\right) = (A_N - K_N C_N) e_N \tag{2.1.5}$$

Equations 2.1.2-2.1.5 can be combined together to be put into matrix form to produce the closed loop system. This matrix is very special in the fact that it enables the stability of the system to be analyzed with the reduced order model based controller incorporated. The closed loop system will be:

$$\begin{bmatrix} \dot{x_N} \\ e_N \\ x_R \\ x_Q \end{bmatrix} = \underbrace{\begin{bmatrix} (A_N + B_N G_N) & B_N G_N & 0 & 0 \\ 0 & (A_N - K_N C_N) & K_N C_R & K_N C_R \\ B_R G_N & B_R G_N & A_R & 0 \\ B_Q G_N & B_Q G_N & 0 & A_Q \end{bmatrix}}_{A_c} \begin{bmatrix} x_N \\ e_N \\ x_R \\ x_Q \end{bmatrix} \tag{2.1.6}$$

The closed loop matrix (A_C) may be stable or unstable. An eigenvalue analysis must be preformed to see where the eigenvalues of A_C lie. The ROM-based controller may have a tendency to force un-modeled modes or residual modes to be unstable. Because something large and complex is being modeled by something very small and simple, the reduced order model may fail at capturing all of the essential dynamics of the large scale system.

If some of these residual modes are in fact forced unstable by the controller, compensation is needed. A residual mode filter (RMF) (5) will be designed to filter out these unwanted instabilities. The unstable residual modes will be designated by the Q subspace which will contain all modes that become unstable in closed-loop; these will be called the *closed-loop instabilities.* The RMF can be formulated such that:

$$\begin{cases} \dot{\hat{x}}_Q = A_Q \hat{x}_Q + B_Q u \\ \hat{y}_Q = C_Q \hat{x}_Q + D_Q u \end{cases} \tag{2.1.7}$$

The RMF filter is attached to the feedback controller and plant as shown in figure 5.

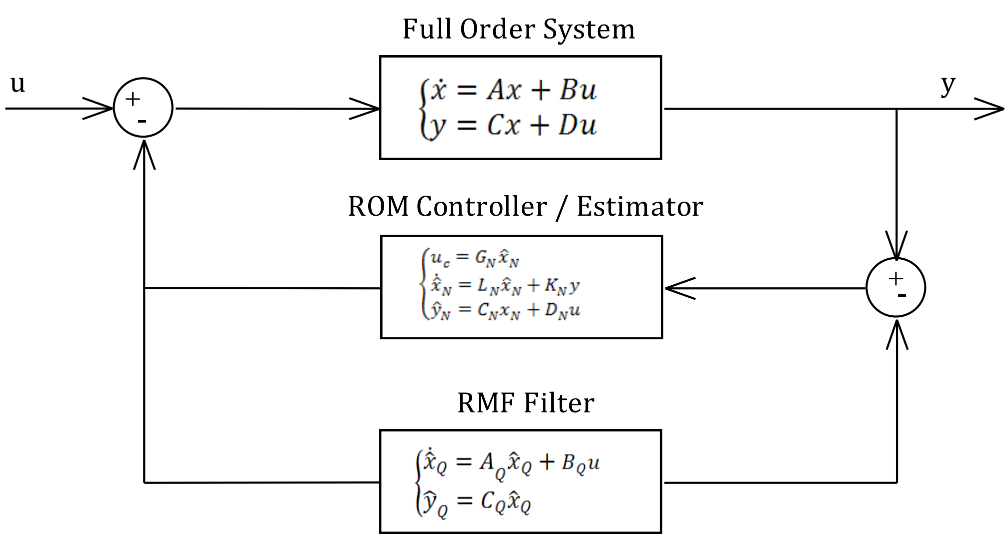

Figure 5: Feedback Control System with RMF Compensation

As before the error term will be introduced and defined as:

$$e_Q \equiv (\hat{x}_Q - x_Q) \tag{2.1.8}$$

Where: $\begin{cases} \dot{e}_Q = A_Q e_Q \\ \hat{y}_Q = C_Q x_Q + C_Q e_Q \end{cases}$

Now, the closed loop system can be derived and is seen in equation (2.1.9).

$$\begin{bmatrix} \dot{x_N} \\ e_N \\ x_R \\ x_Q \\ e_Q \end{bmatrix} = \underbrace{\begin{bmatrix} (A_N + B_N G_N) & B_N G_N & 0 & 0 & 0 \\ 0 & (A_N - K_N C_N) & K_N C_R & 0 & -K_Q C_Q \\ B_R G_N & B_R G_N & A_R & 0 & 0 \\ B_Q G_N & B_Q G_N & 0 & A_Q & 0 \\ 0 & 0 & 0 & 0 & A_Q \end{bmatrix}}_{A_{Crmf}} \begin{bmatrix} x_N \\ e_N \\ x_R \\ x_Q \\ e_Q \end{bmatrix} \quad (2.1.9)$$

This closed loop system is stable. All *open loop instabilities* are contained within the upper left 3 by 3 matrix which was modified to be stable by the reduced order model controller. By definition A_R and A_Q are stable in open loop. Therefore, it is seen all of the interaction of the ROM controller and A_Q has been removed by the RMF, by adding a column of zeroes above the first A_Q and a row of zeroes to the left of the second A_Q.(5)

2.2 Schur Decomposition with RSF Compensation

Schur Decomposition with residual state filtering (RSF) is a reduced order modeling method that is very similar to the previously discussed method, modal truncation with RMF compensation. There are only a few slight differences that will be revealed in the following section. This technique has its advantages and disadvantages from the previous ROM technique.

The system, just as before, is put into its first-order, linearized form as seen in the following equation:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (2.2.1)$$

The A matrix is then transformed into upper diagonal block form using Schur triangulization decomposition method. This method is performed the following way. The A matrix can be decomposed by matrix V such that:

$$A = VA_{schur}V^* \tag{2.2.2}$$

Where V^* is the conjugate transpose of V, and V is also a unitary matrix, therefore, $VV^* = \mathrm{Identity}$. Unitary matrices have some special characteristics: A unitary matrix with all real entries is an orthogonal matrix and satisfies the following inner product: $\langle Vx, Vy \rangle = \langle x, y \rangle$. Consequently, a unitary matrix preserves the norm. Moreover, the columns of V will form an orthonormal basis on the complex space ($\mathbb{C}^n$). Likewise, the rows of V will form an orthonormal basis on $\mathbb{C}^n$. (6)

Matrices in upper triangular block form (Schur form) have some useful advantages over full matrices. Their eigenvalues lie on the diagonal, and may be sorted in any order along the diagonal. If one were controlling an aeroelastic system, he / she might like to organize the eigenvalues such that the slower values (eigenvalues with smaller negative real parts) would appear first along the diagonal, while the faster values (eigenvalues with larger negative real parts) would be near the end of the diagonal. This could be useful to design reduced order models about either the slow or fast eigenvalues of the system.

Once the system is in Schur form, it can be broken up into two subsystems, similar to the modal case:

$$\begin{cases} \dot{x}_r = A_r x_r + A_{rN} x_N + B_r u \\ \dot{x}_N = \qquad\quad A_N x_N + B_N u \\ y = C_r x_r + C_N x_N \end{cases} \tag{2.2.3}$$

The x_N subspace or states will formulate the reduced order model, while the x_r states will be the referred to as the residual states. The x_N states are chosen so that all *open loop instability* is contained within x_N and also contains the most dominant modes for control. Once this system is organized accordingly, a controller and state estimator based on (A_N, B_N, C_N, D) can be designed in the exact same fashion as before.

$$\begin{cases} u_c = G_N \hat{x}_N \\ \dot{\hat{x}}_N = A_N \hat{x}_N + B_N u + K_N (y - \hat{y}_N) \\ \hat{y}_N = C_N x_N + Du \end{cases} \tag{2.2.4}$$

Next, the feedback controller / estimator shown in equation (2.2.4) are attached to the full order system as seen in Figure 4. An error term is again defined as:

$$e_N \equiv (\hat{x}_N - x_N) \tag{2.2.5}$$

The closed loop system can now be formed, so that the overall dynamics / stability of the system can be analyzed. This will look like equation (2.2.6) shown below:

$$A_C \equiv \begin{bmatrix} A_r & A_{rN} + B_r G_N & B_r G_N \\ 0 & A_N + B_N G_N & B_N G_N \\ K_N C_r & 0 & A_N - K_N C_N \end{bmatrix} \quad \text{Where: } \dot{x} = A_C x \tag{2.2.6}$$

The closed loop system (A_C) may be stable or unstable; it all depends on whether the reduced order model controller / estimator forced some un-modeled states to go unstable. If there is no instability the ROM controller has adequately performed its function and no further work is needed! If instabilities do exist, these problematic states must be located and compensated using a residual state filter (RSF). The unstable states are designated as the Q states, and again the system is broken into three subsystems as shown in equation (2.2.7).

$$\begin{cases} \dot{x}_R = A_R x_R + A_{RQ} x_Q + A_{RN} x_N + B_R u \\ \dot{x}_Q = \qquad\qquad A_Q x_Q + A_{QN} x_N + B_Q u \\ \dot{x}_N = \qquad\qquad\qquad\qquad A_N x_N + B_N u \\ y = C_R x_R + C_Q x_Q + C_N x_N \end{cases} \tag{2.2.7}$$

This system is nearly the same as the previous one (equation 2.2.3); the only difference is that the Q states (X_Q) have been separated out of the previous r states (X_r) which produced both X_R and X_Q. This new subsystem will have no affect on the ROM designed controller. The benefit of designating these Q states is that when they are removed from the closed loop system, the system will become completely stable.

Notice how the Schur form reduced order model differs from that of the modal case. The Schur form system has these "spillover blocks" which are the off diagonal blocks connecting the different subspaces. The residual state filter will need to filter out this spillover as well as the *closed loop instability*. The RSF needs to be designed such that:

$$\begin{cases} \dot{\hat{x}}_Q = A_Q \hat{x}_Q + A_{QN} \hat{x}_N + B_Q u \\ \hat{y}_Q = C_Q \hat{x}_Q + D_Q u \end{cases} \tag{2.2.8}$$

Where: $e_Q \equiv \left(\hat{x}_Q - x_Q\right)$ and $\begin{cases} \dot{e}_Q = A_Q e_Q + A_{QN} e_N \\ \hat{y}_Q = C_Q x_Q + C_Q e_Q \end{cases}$

The closed loop system is then formed with the RSF compensation attached. It appears as the following:

$$\frac{d}{dt}\begin{bmatrix} x_R \\ x_N \\ e_N \\ x_Q \\ e_Q \end{bmatrix} = \underbrace{\begin{bmatrix} A_R & A_{RN} + B_R G_N & B_R G_N & A_{RQ} & 0 \\ 0 & A_N + B_N G_N & B_N G_N & 0 & 0 \\ K_N C_R & 0 & A_N - K_N C_R & 0 & -K_N C_R \\ 0 & B_Q G_N & A_{QN} + B_Q G_N & A_Q & 0 \\ 0 & 0 & A_{QN} & 0 & A_Q \end{bmatrix}}_{A_C} \begin{bmatrix} x_R \\ x_N \\ e_N \\ x_Q \\ e_Q \end{bmatrix} \tag{2.2.9}$$

This system (A_C) is not necessarily stable. As it is seen, this matrix differs from the modal truncation with RMF compensation (Equation (2.1.9)) by these spillover blocks, A_{QN} and A_{RQ}. Note that when these matrix blocks are set to zero, this method will be exactly equivalent to the modal truncation method. A_Q will now interact with the ROM system through the matrices A_{QN} and A_{RQ}. This closed loop system can be separated into the following form such that:

$$\underbrace{\begin{bmatrix} A_R & A_{RN}+B_RG_N & B_RG_N & 0 & 0 \\ 0 & A_N+B_NG_N & B_NG_N & 0 & 0 \\ K_NC_R & 0 & A_N-K_NC_R & 0 & -K_NC_R \\ 0 & B_QG_N & A_{QN}+B_QG_N & A_Q & 0 \\ 0 & 0 & 0 & 0 & A_Q \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & A_{RQ} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{QN} & 0 & 0 \end{bmatrix}}_{\Delta A_C}$$

In this form it is simple to see that the spillover (ΔA_C) will affect the stability of the entire system. However, when the norm of ΔA_C ($\|\Delta A_C\|$) is small enough, the stability of A_0will prevail. If the matrix ΔA_C is large enough this interaction between A_Q and the ROM could de-stabilize the system. Further work is needed on this reduced order model approach.

This method is a relatively appealing for formulating and implementing reduced order model control with compensation, because it is computationally trivial to perform the Schur decomposition and also may be used in conjunction with a variety of techniques such as singular perturbations or balanced reduction.

2.3 Singular Perturbation Approach

The third ROM technique is that of the small perturbation approach. This method is quite different from the two techniques previously discussed. The singular perturbation approach utilizes the different speeds of the plant's eigenvalues. The speed of an eigenvalue is determined by the magnitude of the negative real part. Slower eigenvalues will have smaller magnitudes, while faster eigenvalues will have larger magnitudes in relation to each other. That is, the system may be broken up into slow and fast modes / states for the design of the reduced order model controller. In the case of aeroelastic feedback control, the fluid states' reaction times will be much quicker than the structure's, so the fluid states will be the *faster* states. This means that the real parts of the eigenvalues will be consistently larger than the structural eigenvalues, which is verified by the plot of poles in Figure 2. These features will be discussed in further detail later in this section.

Just as before, the system is put into the first order linear form as seen in equation (2.3.1):

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.3.1}$$

The primary goal of this approach is to break up the vector of states (x) such that the slow states are separated from the faster states. For aero-elastic models, plants which contain fluid and structural states, the faster states will obviously be those of the fluid; this means basically that the fluid will react much quicker than the structure, which is true from physics. If the system is broken up into structure / fluid blocks it looks like the following:

$$\begin{cases} \dot{x}_s = A_s x_s + A_{sf} x_f + B_s u \\ \dot{x}_f = A_f x_f + A_{fs} x_s + B_f u \\ \dot{y} = C_f x_f + C_s x_s \end{cases} \tag{2.3.2}$$

For aero-elastic models breaking the system into structure/fluid subspaces is a simple task to do. The fluid states are completely distinct from the structure states, so these matrices are easily formed. Next, a small perturbation parameter (ε) is introduced to modify the fluid system, as seen in equation (2.3.3).

$$\begin{cases} \dot{x}_s = A_s x_s + A_{sf} x_f + B_s u \\ \varepsilon \dot{x}_f = A_f x_f + A_{fs} x_s + B_f u \\ \dot{y} = C_f x_f + C_s x_s \end{cases} \tag{2.3.3}$$

The reduced order model is then designed such that $\varepsilon = 0$. Epsilon equal to zero states that the fluid states will be infinitely fast when compared to the structural states. Since the derivative of the fluid will now be zero, this implies that the fluid will react to external forces or pressures immediately. This seems like a strong assumption to make, but if the system one is modeling is an aircraft flying at mach 1 speeds, this will be undoubtedly true. The timescale of the fluid will be much smaller than the structure. Once the fluid system is modified by the singular perturbation, the equation will look like:

$$0 = A_f x_f + A_{fs} x_s + B_f u \tag{2.3.4}$$

Next, if A_f is completely stable and invertible then the fluid states x_f can be solved for, yielding:

$$x_f = A_f^{-1} A_{fs} x_s + A_f^{-1} B_f u \tag{2.3.5}$$

The fluid states can then be plugged into the rest of the plant model to get the following result:

$$\begin{cases} \dot{x}_s = \overbrace{\left(A_s - A_{sf}A_f^{-1}A_{fs}\right)}^{A_N} x_s + \overbrace{\left(B_s - A_{sf}A_f^{-1}B_f\right)}^{B_N} u \\ \dot{y} = \underbrace{\left(C_s - C_f A_f^{-1}A_{fs}\right)}_{C_N} x_s + \underbrace{\left(-C_f A_f^{-1}B_f\right)}_{D} u \end{cases} \tag{2.3.6}$$

The reduced order model is then represented by the matrices A_N, B_N, C_N, and D. This technique reduces the size of the system to the size of the structural system by inserting this perturbation (ε) which approximates the fluid as having an infinitely fast derivative. If further reduction is still needed at this point, another combination of methods may be used. For example, balanced reduction may be applied, or even the two previously discussed methods may be used. The ROM feedback controller / estimator are then designed to form the following matrices: (4)

$$\begin{cases} u_c = G_N \hat{x}_N \\ \dot{\hat{x}}_N = A_N \hat{x}_N + B_N u + K_N (y - \hat{y}_N) \\ \hat{y}_N = C_N x_N + Du \end{cases} \tag{2.3.7}$$

Again the stability of the entire system needs to be analyzed to see how well the reduced order model estimated the full scale model. The easiest way to do this is formulate the closed loop system which can be seen in the following equation:

$$\frac{d}{dt}\begin{bmatrix} x_s \\ x_f \\ e_N \end{bmatrix} = \underbrace{\begin{bmatrix} A_s + B_s G_N & 0 & B_s G_N \\ \frac{1}{\varepsilon} A_{fs} & \frac{1}{\varepsilon} A_f & 0 \\ A_{sf} & 0 & A_s - K_N C_s \end{bmatrix}}_{A_c} \begin{bmatrix} x_s \\ x_f \\ e_N \end{bmatrix} \tag{2.3.8}$$

There is a theorem by Balas that states: (7)

$$\exists\, \varepsilon_* > 0 \;\; \forall\, 0 \le \varepsilon \le \varepsilon_* \;\ni\; A_c \; stable$$

According to this theorem there exists a range of singular perturbations (ε) for which this system (Equation 2.3.8) will be stable. The size of this upper bound ε_* is unknown, but can be estimated as in (7). It may be close to one or may quite small.

In aeroelastic systems the singular perturbation parameter can be directly related to the inverse of the fluid flow velocity, $v = \frac{1}{\varepsilon}$. As the velocity grows the singular perturbation tends towards zero. This states that at extremely high velocity values epsilon will go to zero, thus validating the idea that the fluid is *infinitely fast* when compared to the structure.

3 Results

In this section the results for the three previous methods for reduced order modeling will be discussed. Each technique was applied to an aero-elastic model of the F-16 aircraft. Rather different results were observed for each of the methods.

3.1 Modal Truncation

The modal truncation method, described Section 2.1, was implemented and the following results were obtained. The controller / state estimator were designed such that the eigenvalues of the matrices (A_N+$B_N G_N$) and (A_N-$K_N C_N$) were all stable and were placed in fast enough positions to properly control the system. These eigenvalues can be seen in the following figure. The estimator eigenvalues were designed to be about 4 times faster than the controller.

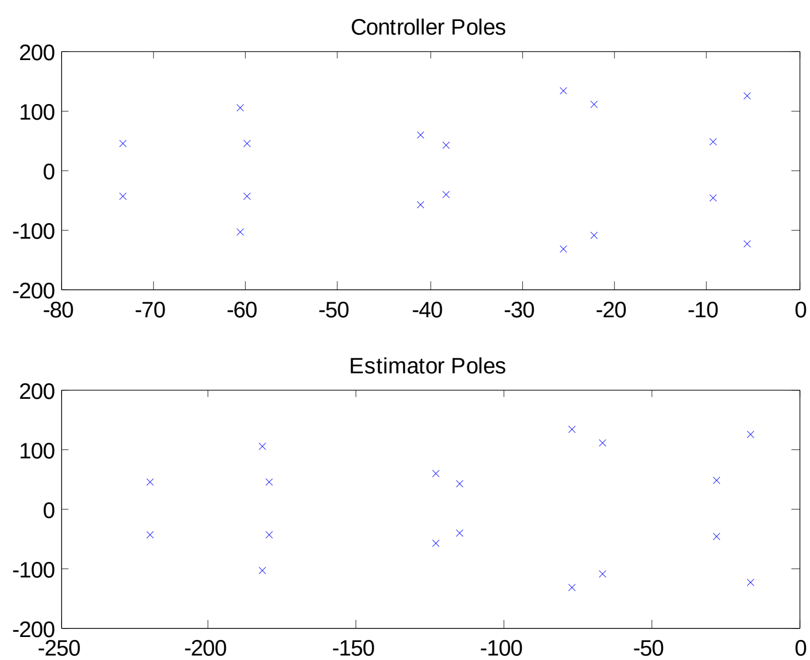

Figure 6: Controller and Estimator Pole Design

The closed loop eigenvalues were plotted in Figure 7. Notice that one unstable mode (two unstable states) exists. These were found to be states 50 and 51 in the fluid system. These are the states where the ROM controller had an adverse affect. A residual mode filter must be used to compensate for these problematic states.

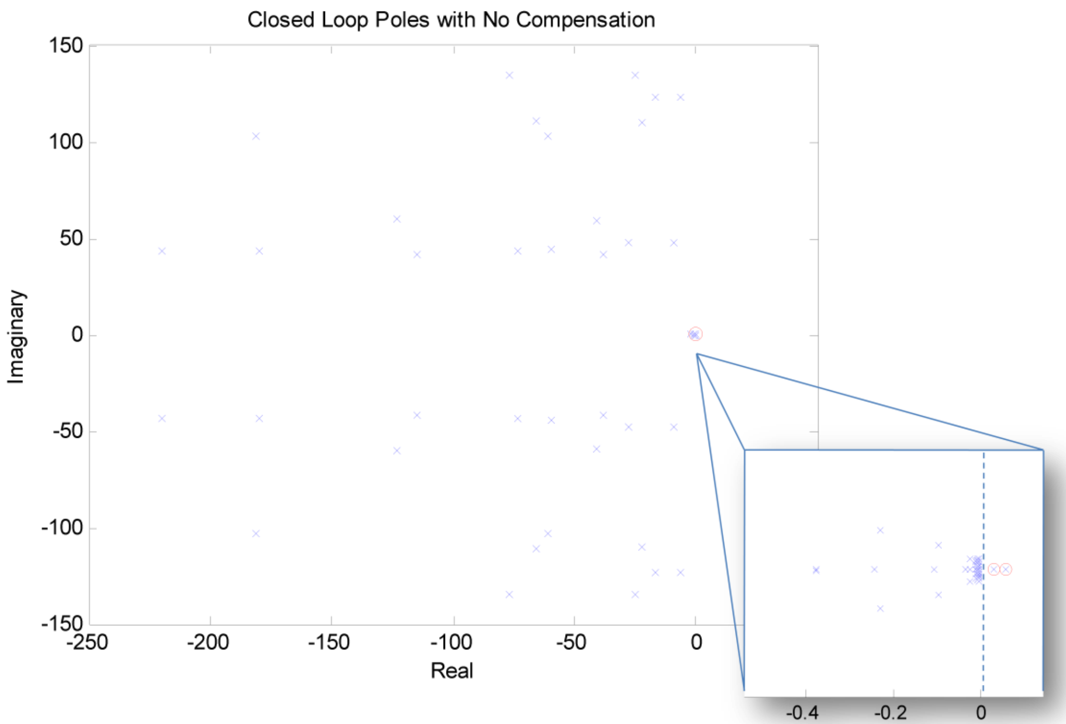

Figure 7: Closed Loop Poles of Equation 2.1.6

One may think that these two states are insignificant and would not affect the dynamics of the entire system, but indeed they do. Any instability in a system is always a problem and can never be neglected.

Simulink was then utilized to simulate the performance of the system over 5 seconds. As seen in Figures 8 and 9 states 50 and 51 of the fluid system drive the entire system completely unstable; that is the outputs and states go toward infinity as time increases.

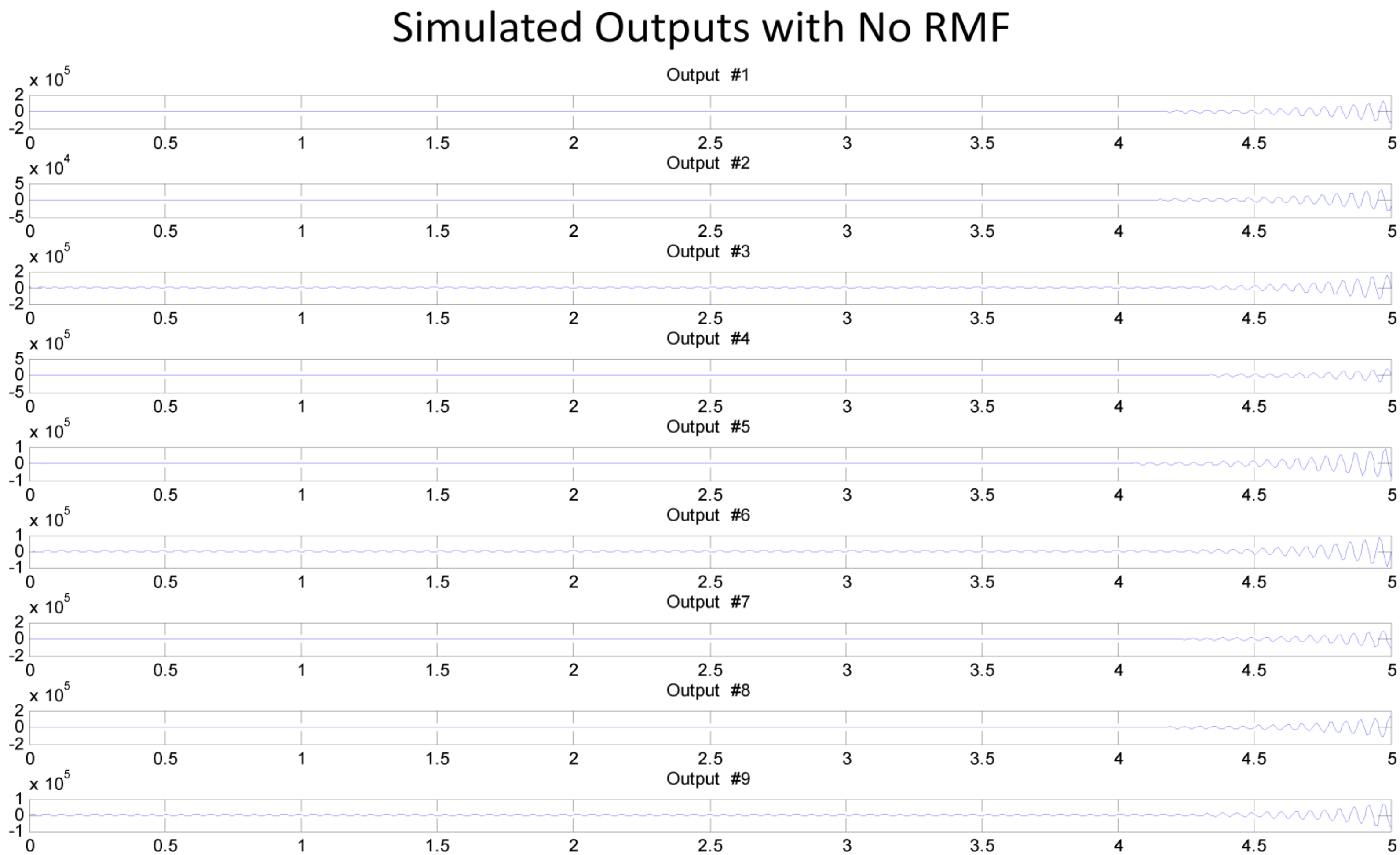

Figure 8: Outputs (y) of System with No Compensation

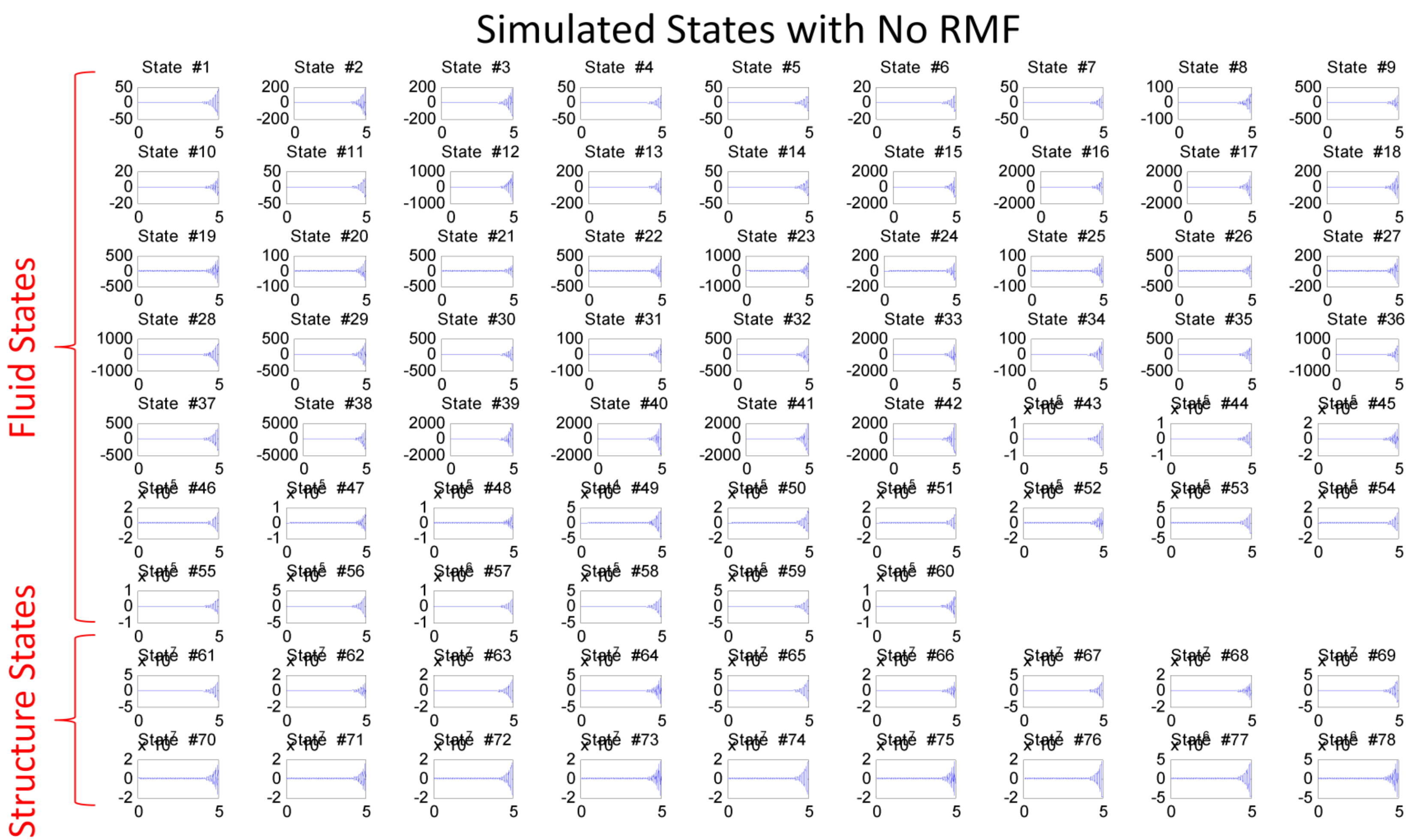

Figure 9: States (x) of System with No Compensation

Once it was determined that the unstable states were the fluid states [50, 51], these were separated into the x_Q subsystem. The RMF was designed on these states. When implemented, the closed loop system in equation (2.1.9) was created. The eigenvalues of the matrix A_{Crmf} were plotted in Figure 10.

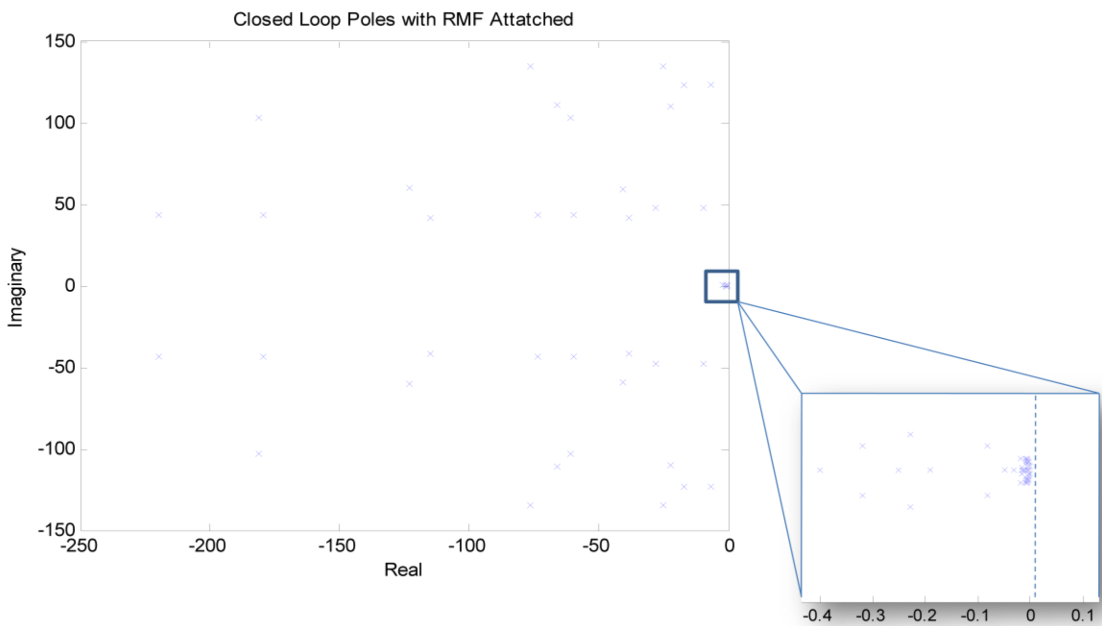

Figure 10: Closed Loop Poles of Equation 2.1.9

It can be seen that the RMF has done its job and stabilized those problematic states. All modes now lie on the left half plane, so the goal of the compensator was achieved: A residual mode filter was used to stabilize a modal truncation ROM of an aeroelastic system. Again Simulink was used to analyze the performance of the system over time. Figures 11 and 12 show that the system's states and outputs now converge to zero. If design constraints existed, for example if the system needed to converge to zero within 0.1 seconds, the control designer would then have to go back and design a new controller that would do this, but the RMF would remain the same.

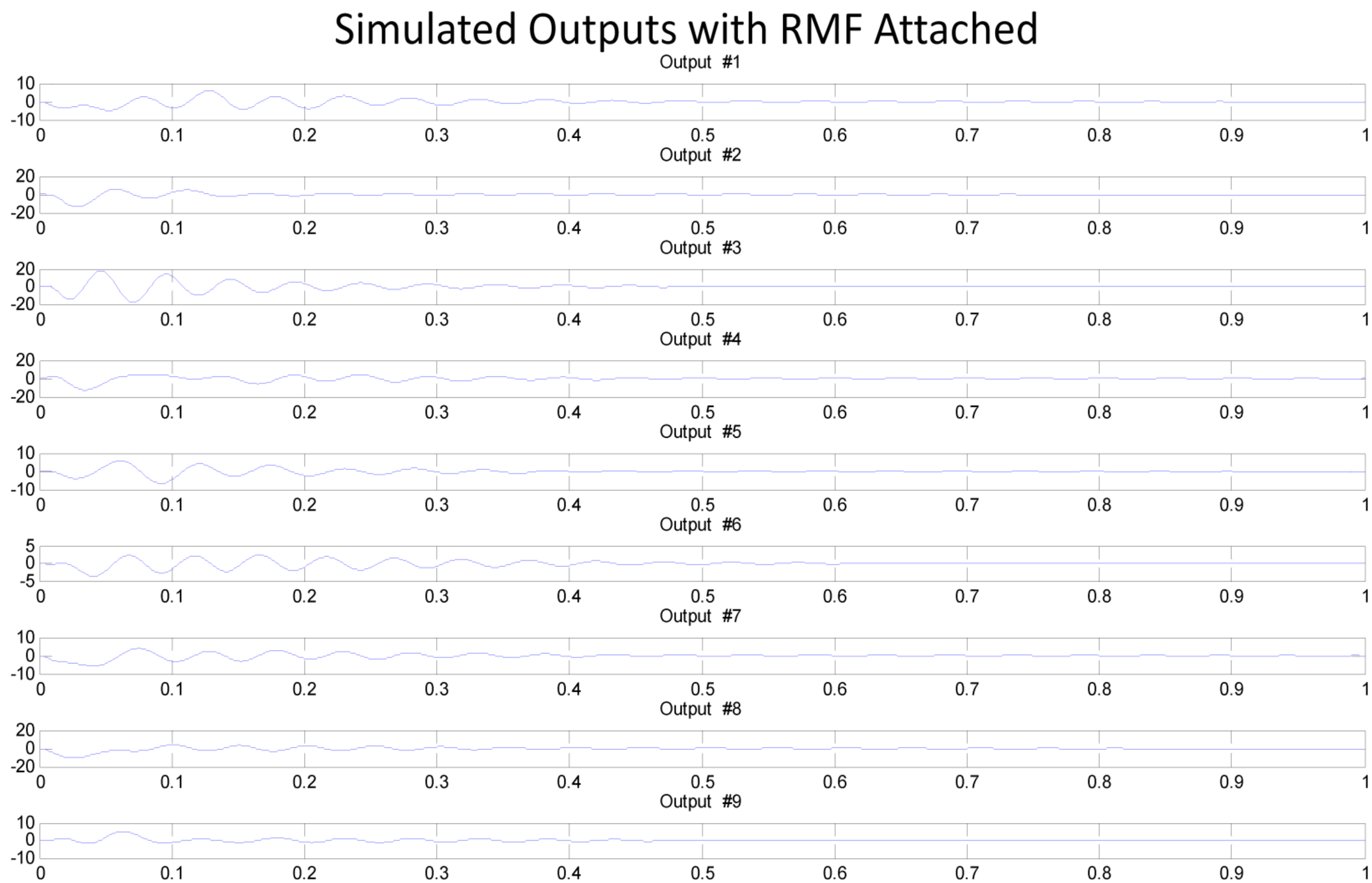

Figure 11: Outputs (y) of System with Compensation

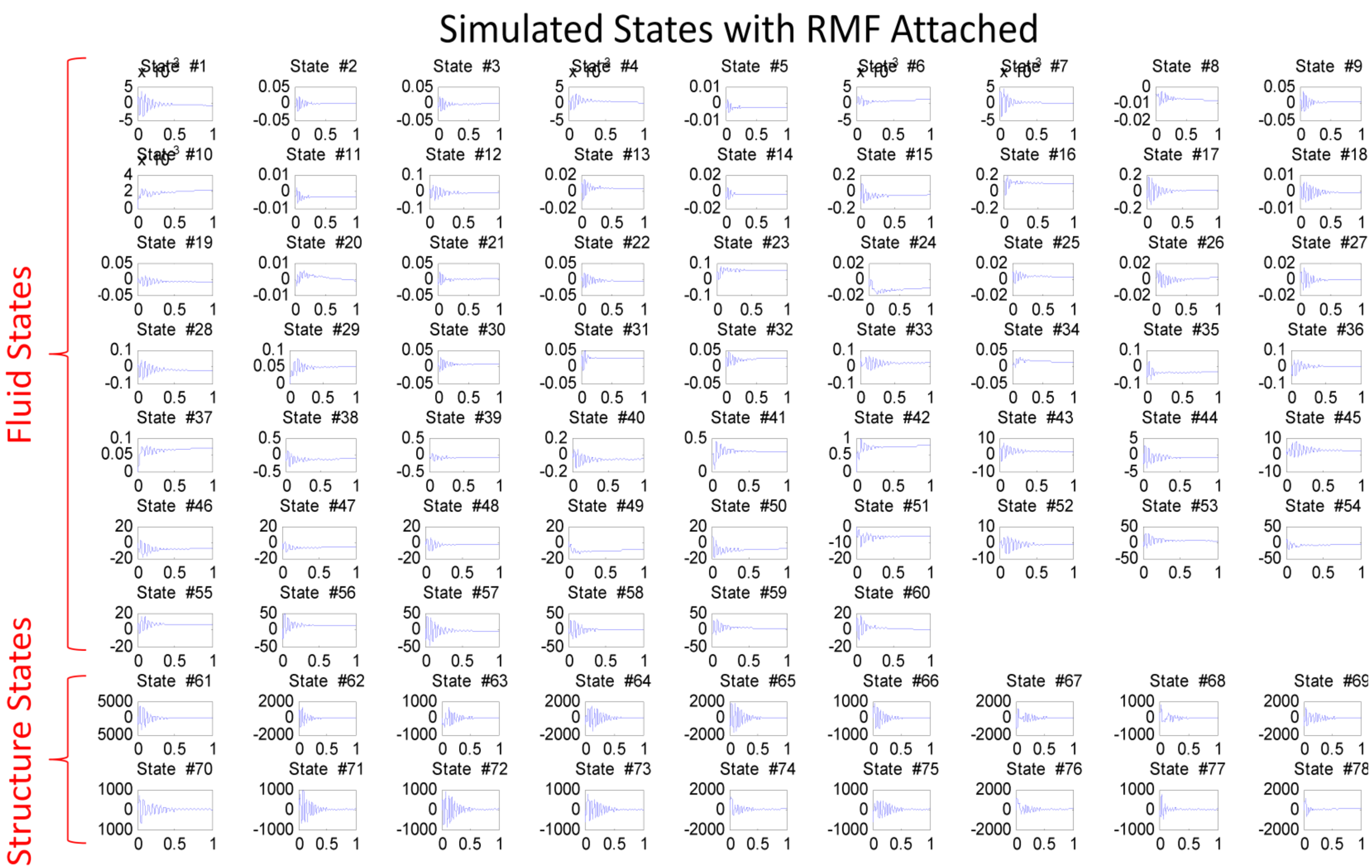

Figure 12: States (x) of System with Compensation

3.2 Schur Form Results

The method in Section 2.2 was followed and the results are presented here. The controller was designed such that the poles of the controller / state estimator (A_N+$B_N$$G_N$) and ($A_N$-$K_N$$C_N$) were stable. This is verified figure 13. Again the estimator poles were designed to be approximately 4 times faster than the controller. The speed of the controller eigenvalues were chosen by trial and error. They needed to be much more stable than the eigenvalues of the plant to produce good response.

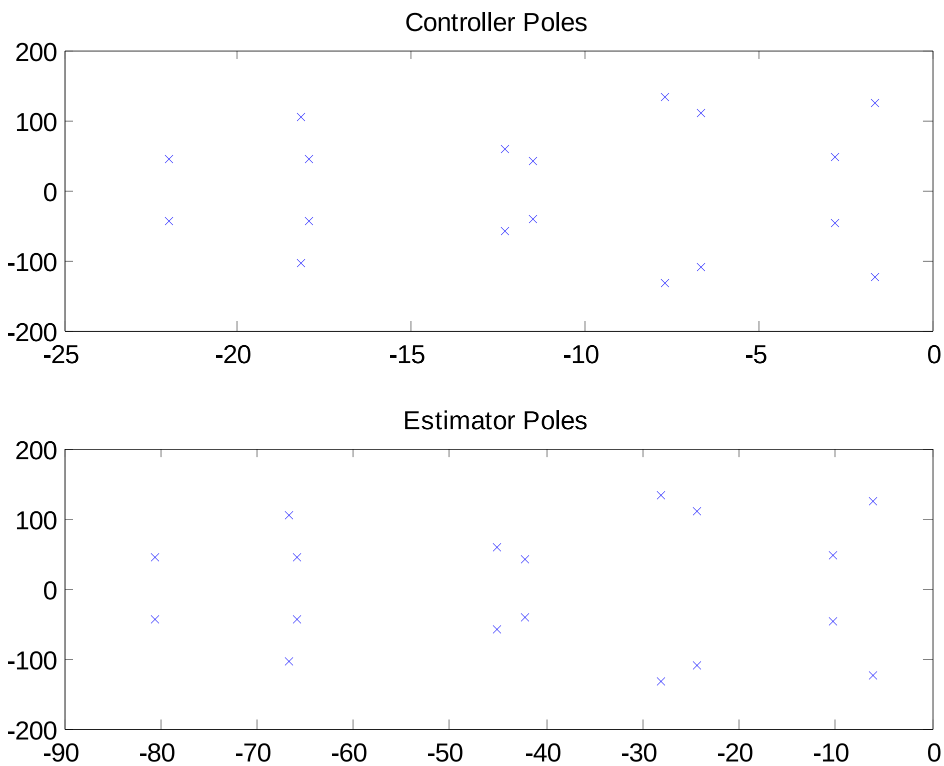

Figure 13: Real and imaginary plot of Controller / Estimator Poles of the Schur Form ROM based Controller

Once the controller was designed and implemented, the closed loop system was formed as seen in equation 2.2.9. This closed loop system is very similar to the modal closed loop system, except now the ROM based controller is designed about a non-diagonal matrix. The Schur form closed loop system is seen in Figure 14.

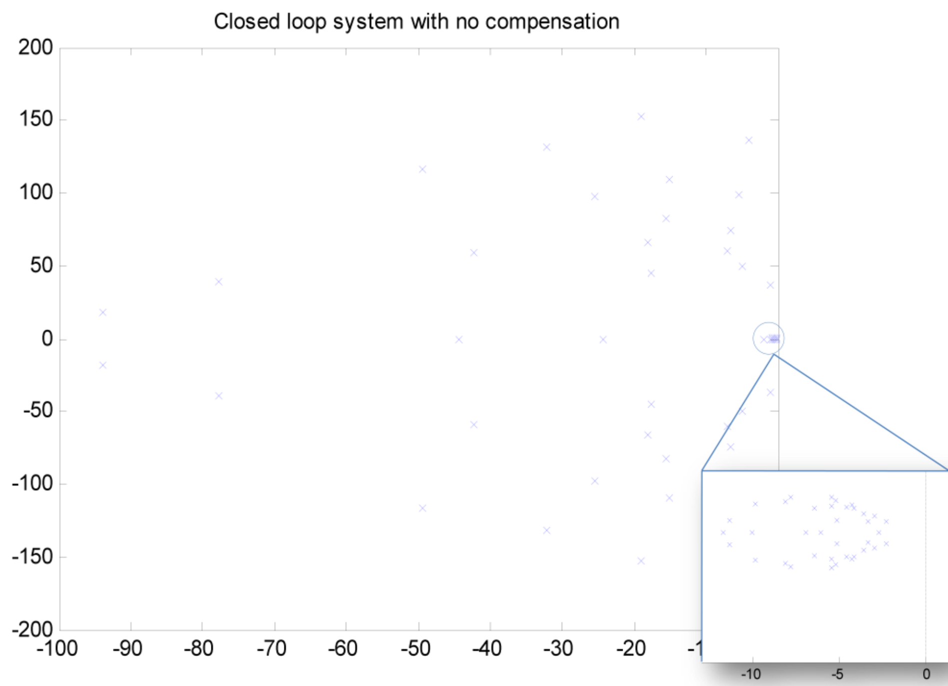

Figure 14: Schur Form Closed Loop System Eq (2.2.9)

The closed loop system is completely stable; all eigenvalues lie in the left half plane. Because the ROM based feedback controller did not cause any un-modeled states to become instable, no compensation was needed here! If the controller had forced states to be unstable a residual state filter would be needed to filter out those adverse affects. Simulink was used to verify the results of the closed loop system. A simulation was performed over a sample time of 0.5 seconds to analyze the performance of the system; this is presented in Figure 15 and 16.

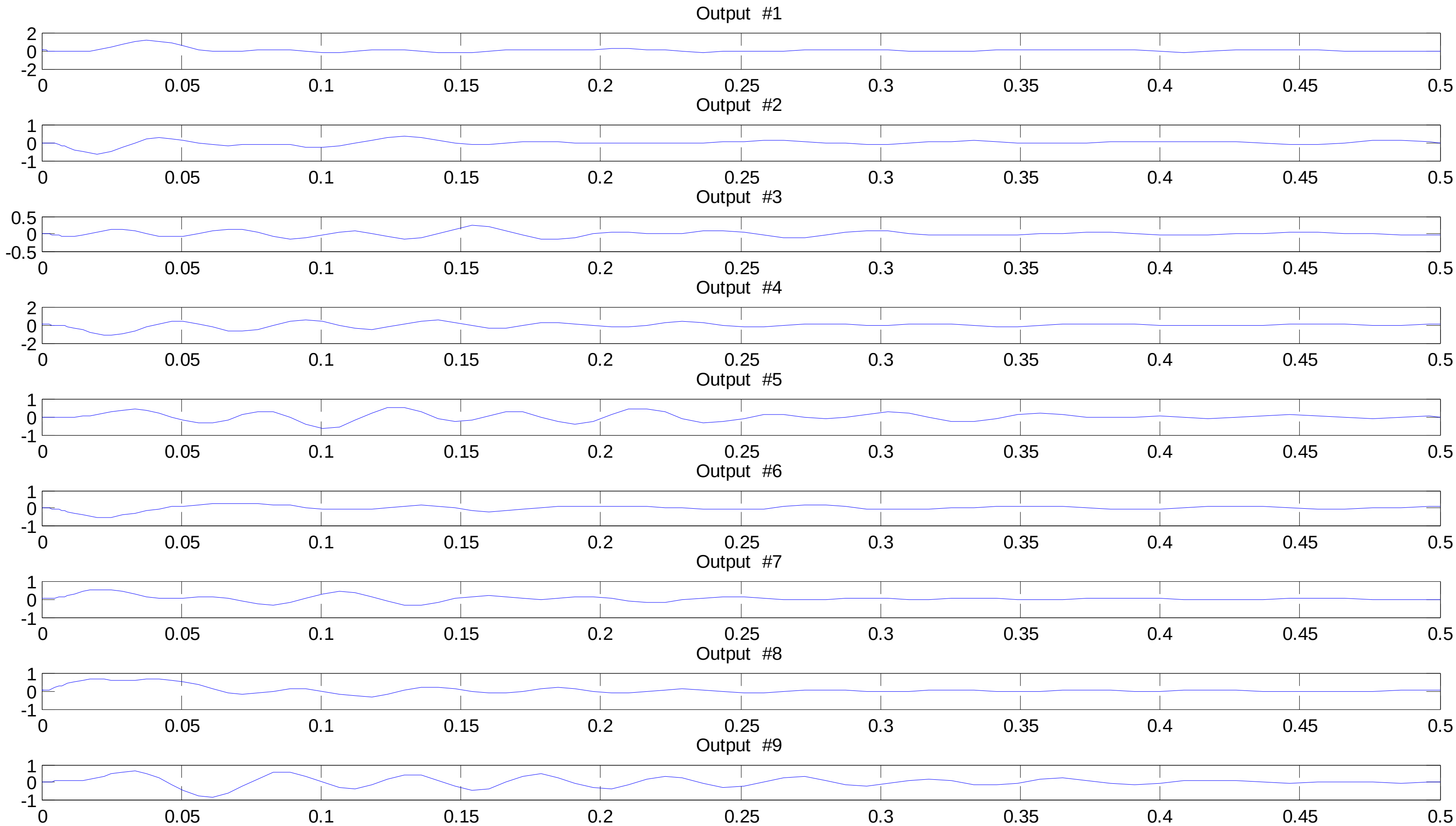

Figure 15: Outputs (y) of system in Schur form No Compensation

Figure 16: States (x) of system without compensation

As seen in figures 15 and 16, the system does converge to zero in a relatively quick time. The states in Figure 16 tend to oscillate more when compared to the modal case. This indicates that indeed spillover is occurring here. The states are now influencing each other through the off diagonal blocks in the upper triangular block form.

3.3 Singular Perturbation Results

The method for singular perturbation parameter, laid out in Section 2.3, was followed and the following results were obtained. The simulink plots in the following figures show the different simulations with a certain size perturbation. An upper bound on the singular perturbation was found.

This upper bound (ε_*) was found to be approximately 0.00074. At epsilons lower than 0.00074 the closed loop system seen in equation (2.3.8) is completely stable. Once the perturbation grew above this ε_*, the system would go unstable. Therefore there exists an upper bound for which: (8)

$0 \leq \varepsilon \leq \varepsilon_* \; where \; \varepsilon_* = 0.00074$. for A_C Stable

This is seen in the following figures. The significance of this number is not known at this point. It *suggests* that this ROM based control system may only work when the system has velocities related to $\frac{1}{\varepsilon_*}$. If this is the case, $\frac{1}{\varepsilon_*} = 1351$ (ft/s) would be an accurate description of supersonic speeds. Although the units are unknown in this model, it is hard to decide without communication with the author of the F-16 model (Charbel Farhat) what the significance of this number really is.

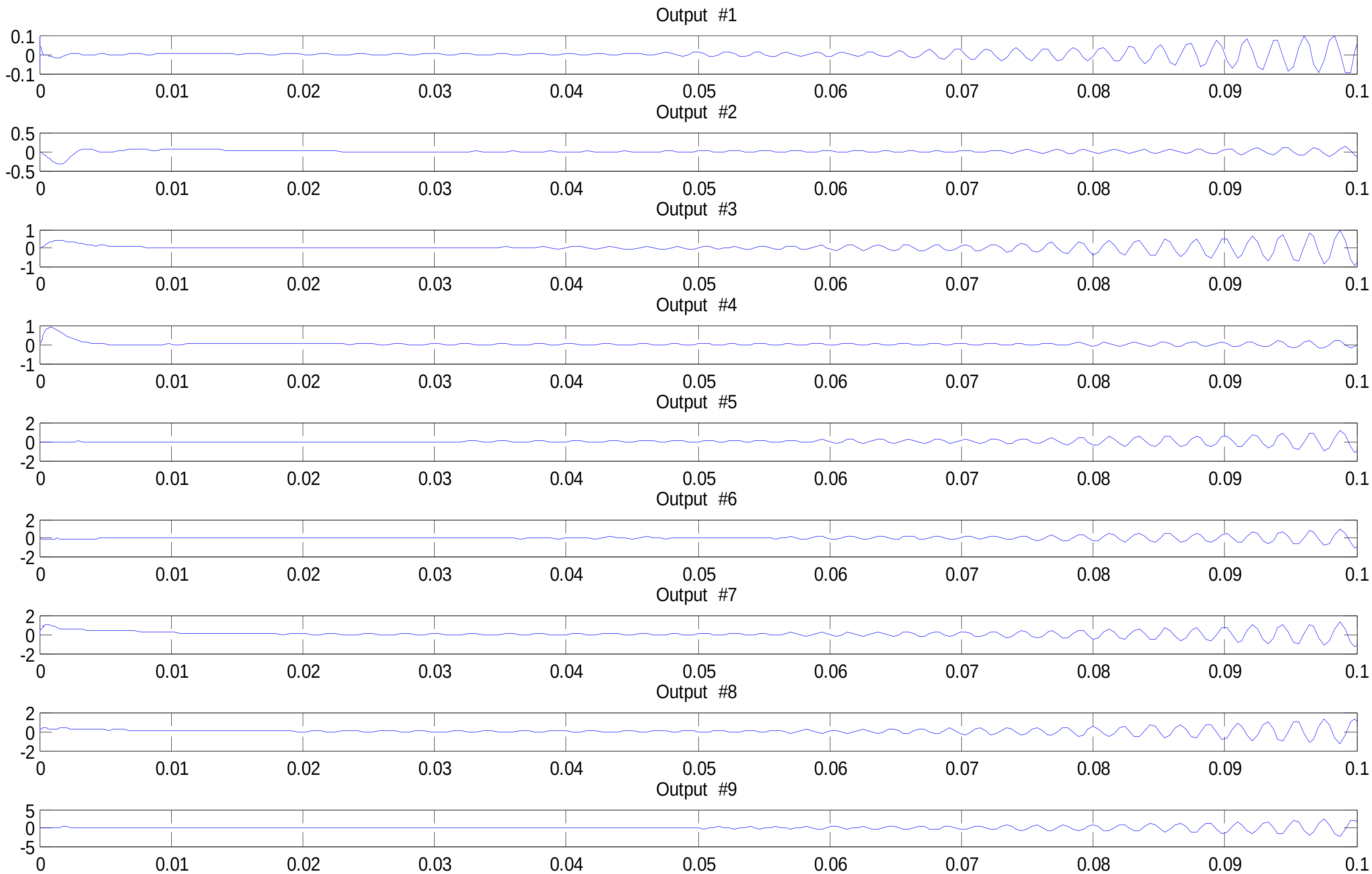

Figure 17: Outputs of System with ε > ε*

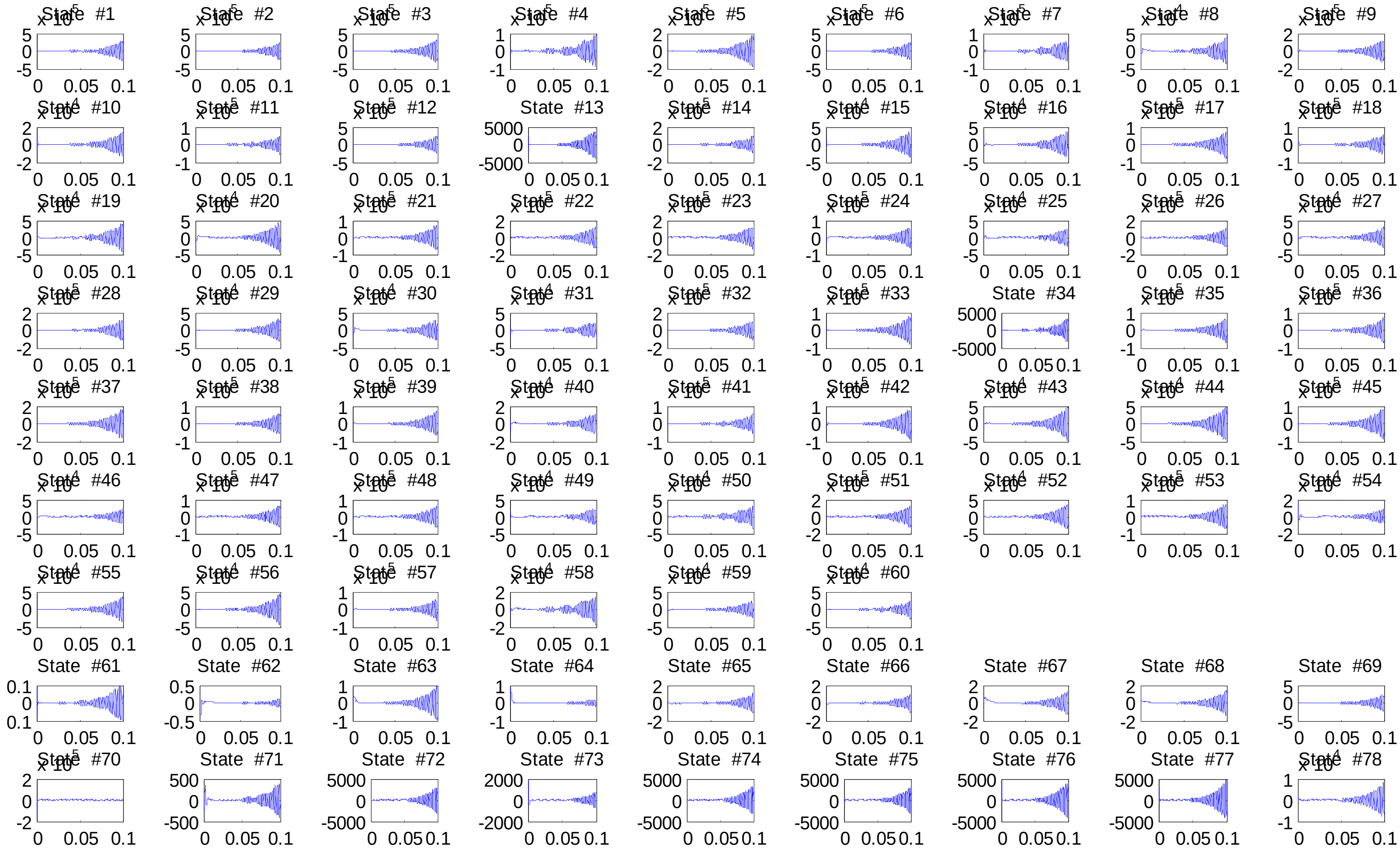

Figure 18: States of System with $\varepsilon > \varepsilon_*$

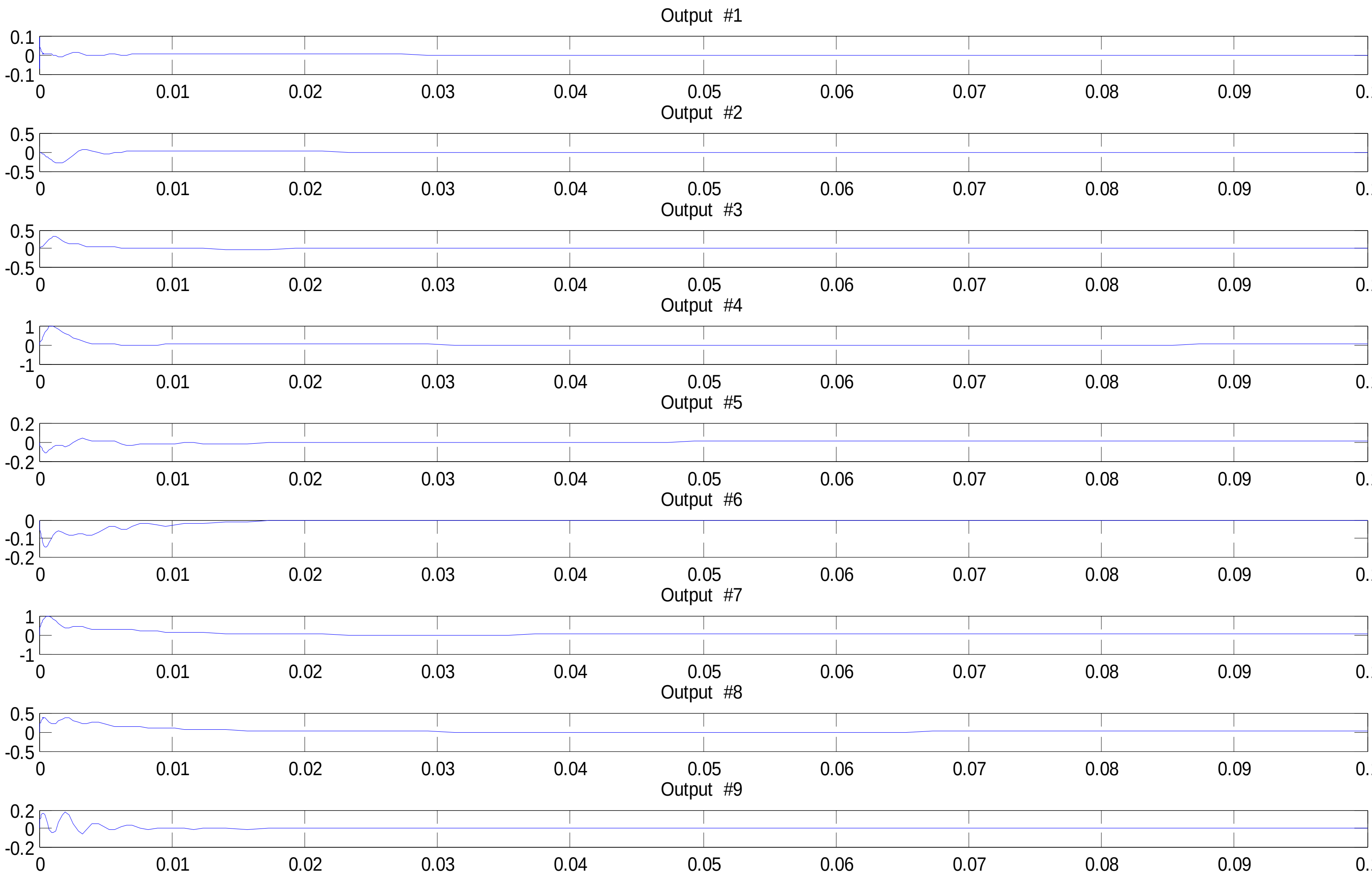

Figure 19: Outputs of System with ε < ε*

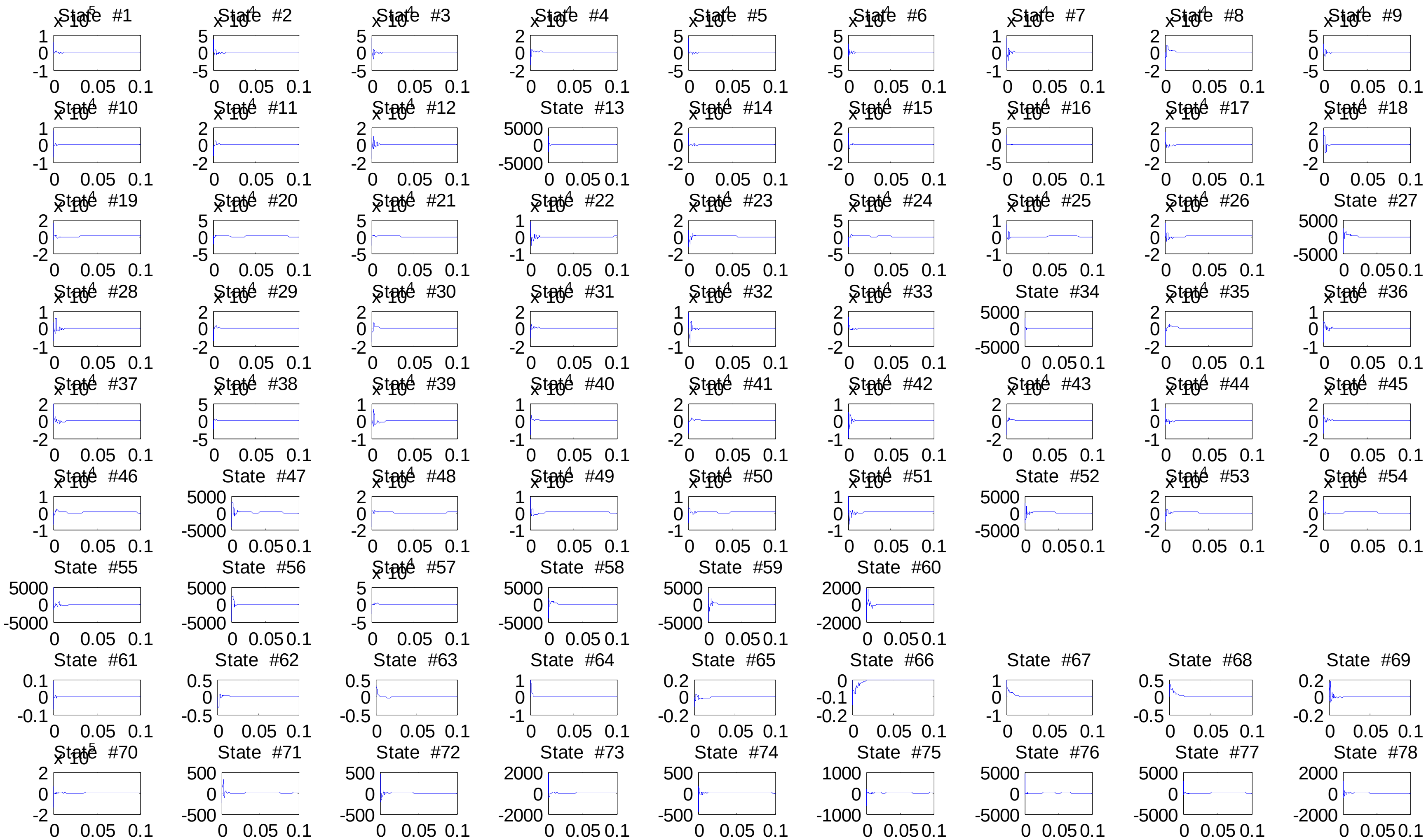

Figure 20: Outputs of System with $\varepsilon < \varepsilon_*$

4 Discussion

In summary, the three reduced order model methods for implementing feedback control were carried out on the F-16 aero-elastic model and the results have been shown. The full order model consists of 78 states; the reduced order model for the three techniques is 18 states, approximately a 75% reduction. Further reduced order models tended to produce more potential instability in the system. Each method has its advantages and disadvantages. The following section will explore the pros and cons of each of the previously presented ROM techniques.

The modal truncation method provides an excellent means for designing ROM based controllers for structures only. The modes of the structure are very well distinguished; the eigenvalues are largely separated and very sparse, which allows for an easy choice to retain or throw away certain eigenvalues. The structural modes are also directly related to the resonant frequencies of the structure. The lower frequency modes have the highest displacements and highest energy states. One can see that these lower frequency modes would be the best to capture, while the higher frequency modes would be the ones to neglect. In fact, designing the ROM about the lowest modes in a structure is a very good representation for the dynamics of entire system.

This method has not been applied to structure-fluid systems, until now. Fluid systems' eigenvalues tend to be very closely positioned which makes it difficult to eliminate certain modes while retaining others. It is also difficult to determine the most dominant modes in the fluid system. Much more fluid dynamics knowledge is needed for the model to determine which modes are needed and which can be neglected.

In the F-16 model, all fluid modes were thrown away and the ROM was only designed about the structural modes. The feedback controller was designed from that information and then implemented on the full order model. After the designed controller was implemented on the full model, instabilities were produced in the fluid system. At this point a residual mode filter was used to filter out this instability in the fluid. It is important to note that one could have gone about this ROM completely differently. The designer could have chosen to design the ROM about only the fluid modes, or both the fluid and structure. These ideas should be considered in a future project.

The Schur form method for model reduction is a very appropriate method to choose for aeroelastic models. Upper triangular block form allows for certain advantages over the modal case. If the scale of the system is on the order of 10^5 states, diagonalization is a costly method. Putting the system into upper triangular block form should be easier and have less computational errors. Although no residual state filter was needed in this experiment, if it was required the RSF would perform similarly to the RMF compensation in the modal case. However, the RSF will be a bit more complex because it will now need to contain the spillover blocks.

The singular perturbation approach allows for the separation of the system into slow and fast states. This perturbation approach is seen to be extremely useful in the aeroelastic areas because of the large differences in structure and fluid timescales. The singular perturbation may also be related to a velocity term, which is an extremely useful observation. This approach can also be used in conjunction with the other two methods to produce ROM based controllers.

Different combinations of these methods may be a good area for future work. For example, Schur Decomposition method used along side with the single perturbation parameter and RSF compensation has promising features. This hybrid method would allow for extremely large reductions with highly accurate results. This would be another area for future research.

Also, balanced reduction would be an alternative combination to use with the single perturbation parameter. Balanced reduction is an algorithm that calculates the Hinkel values of the linear time-invariant system. The Hinkel values determine the most controllable and observable states in the system, so that the ROM can be based of of those. Balanced reduction optimizes controllability and observability in a LTI system. Although no balanced reduction was used in this research project, it may turn out to be a very promising alternative to reduced order model based control of aeroelastic simulations.

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